Fast simulation for slow paths in Markov models

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Introduction

Consider a *Continuous-time* Markov chain (CTMC)

Event of interest:
From a given initial state, reach some target state(s) "on time"

"On time" may mean:

(i) *before* some time bound τ

(ii) *after* some time bound τ

τ is rarity parameter:

$$\mathbb{P}(\text{event}) \downarrow 0 \text{ as } \tau \downarrow 0 \text{ resp. } \tau \to \infty$$

Our focus: case (ii)
Goal: estimate this probability using Importance Sampling (IS).

Motivation

- Model checking stochastic systems: $\mathbb{P}$(failure) below threshold?
- Often large state spaces: numerical techniques not applicable.
- Solution: statistical model checking (=simulation)
- When ‘failure’ is rare:

Motivation

- Model checking stochastic systems: $\mathbb{P}$(failure) below threshold?
- Often large state spaces: numerical techniques not applicable.
- Solution: statistical model checking (=simulation)
- When ‘failure’ is rare: we’re in business!

But why focus on reaching target state *after* some (large) time bound?

- In Markov reward model ‘failure’ can e.g. mean ‘collect too much reward/cost before absorption’
- This event is equivalent with ‘absorption after sufficiently large time’ in a related CTMC.

Single path

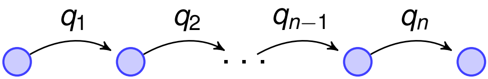
- We study a single path, i.e. pure birth process

$$\begin{tikzcd} \bigcirc \arrow[r, bend left, "q_1"] & \bigcirc \arrow[r, bend left, "q_2"] & \cdots \arrow[r, bend left, "q_{n-1}"] & \bigcirc \arrow[r, bend left, "q_n"] & \bigcirc \end{tikzcd}$$

  Rates q_j are general (need not be different)
- Interesting on its own: sum of ~~i.i.d.~~ r.v.'s grows large. (but here n is fixed)
- Also needed for two-step approach in general CTMC:
  - First select appropriate paths
    (How? Open question...)
  - Then consider each path separately

Model and goal

- We consider a pure birth process on $\{1, \ldots, n+1\}$
- Initial state is 1, target state is $n+1$.
- Sojourn times are $T_j, j = 1, \ldots, n$
- T_j are independent, densitiy $f_j(t) = q_j e^{q_j t}$
- $T = \sum_{j=1}^{n} T_j$



Interest: estimate $\mathbb{P}(T > \tau)$ for large τ, using IS
How to find a good change of measure?

- Time-dependent, forcing T to meet the time bound? No...
- Replace densities by approximations of conditional densities, given $T > \tau$

Monte Carlo / Importance sampling

- Monte Carlo estimation, N simulation runs,
  t_{ij} is realized sojourn time in state j during run i
- Standard MC estimator for $\mathbb{P}(T > \tau)$:
  (t_{ij} sampled using $f_j(t)$)

$$\hat{p} = \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{\sum_j t_{ij} > \tau},$$

- IS estimator for $\mathbb{P}(T > \tau)$:
  (t_{ij} sampled using $f_j^*(t)$)

$$\hat{p}^* = \frac{1}{N} \sum_{i=1}^{N} \prod_{j=1}^{n} \frac{f_j(t_{ij})}{f_j^*(t_{ij})} \mathbf{1}_{\sum_j t_{ij} > \tau},$$

- If $f_j^*(t)$ equals $f_j(t | T > \tau)$: zero variance
  Analyze conditional behavior of T_j, given $T > \tau$

Conditional behavior of T_j, given $T < \tau$

First consider case (i) for comparison

- Observation:
  For T to be small, all T_j need to be small
- Outcome:

As $\tau \downarrow 0$, 'burden' of small T is shared ~~proportionally~~ by all T_j

Conditional behavior of T_j, given $T < \tau$

First consider case (i) for comparison

- Observation:
  For T to be small, all T_j need to be small
- Outcome:

As $\tau \downarrow 0$, 'burden' of small T is shared *equally* by all T_j

(in fact, T_j are jointly uniform on 'triangle' $\sum_j T_j \leq \tau$)

Conclusion for case (i)

- Replace conditional density $f_j(t|T < \tau)$ of T_j by

$$\frac{n}{\tau}\left(1 - \frac{t}{\tau}\right)^{n-1}, \qquad 0 < t < \tau,$$

  for all $j = 1, \ldots, n$
- $E(T_j|T < \tau) \sim \tau/(n+1)$, so $E(T|T < \tau) \sim \frac{n}{n+1}\tau$
- Also works for non-exponential densities $f_j(t)$

Conditional behavior of T_j, given $T > \tau$

Back to our focus, case (ii)

- Observation:
  For T to be large, not all T_j need to be large!
- Outcome:

As $\tau \to \infty$, 'burden' of large T is shared ~~proportionally~~ by all T_j

Conditional behavior of T_j, given $T > \tau$

Back to our focus, case (ii)

- Observation:
  For T to be large, not all T_j need to be large!
- Outcome:

As $\tau \to \infty$, 'burden' of large T is shared *dis*proportionally by T_j

Conditional analysis

- In general,

$$\mathbb{P}(T_1 > t | T > \tau) = \int_t^{\infty} \frac{f_1(t_1)}{\mathbb{P}(T > \tau)} \mathbb{P}(T - T_1 > \tau - t_1)\, dt_1$$

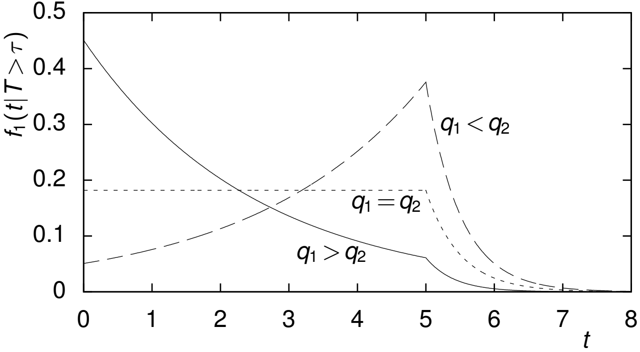
hence

$$f_1(t | T > \tau) = \begin{cases} \dfrac{f_1(t)}{\mathbb{P}(T > \tau)} \mathbb{P}(T - T_1 > \tau - t) & \text{if } t < \tau, \\ \dfrac{f_1(t)}{\mathbb{P}(T > \tau)} & \text{otherwise.} \end{cases}$$

- For $n = 2$,

$$f_1(t | T > \tau) = \begin{cases} \dfrac{q_1 e^{-(q_1 - q_2)t}}{\frac{q_1}{q_1 - q_2} + \frac{q_2}{q_2 - q_1} e^{-(q_1 - q_2)\tau}} & \text{if } t < \tau, \\ \dfrac{q_1 e^{-q_1 t}}{\frac{q_1}{q_1 - q_2} e^{-q_2 \tau} + \frac{q_2}{q_2 - q_1} e^{-q_1 \tau}} & \text{otherwise.} \end{cases}$$

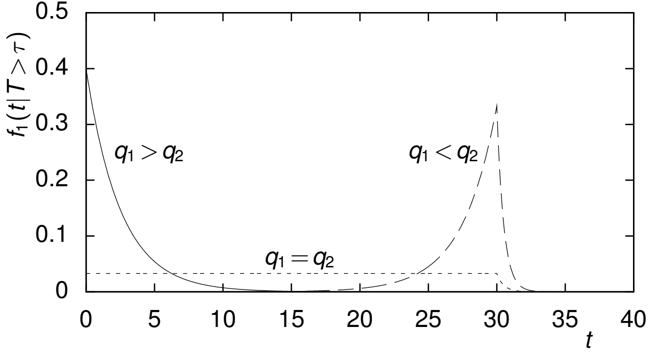
Conditional analysis, $n = 2$



$f_1(t|T > \tau)$ for $\tau = 5$

Exponential rate q_1 for $t > \tau$; $q_1 - q_2$ for $t < \tau$.

Conditional analysis, $n = 2$

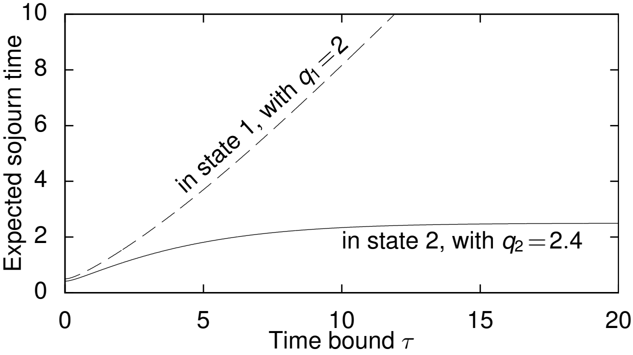


$f_1(t|T > \tau)$ for $\tau = 30$

Exponential rate q_1 for $t > \tau$; $q_1 - q_2$ for $t < \tau$.

Expected share of the burden, $n = 2$

$$\mathbb{E}(T_1 | T > \tau) \sim \begin{cases} \tau & \text{if } q_1 < q_2 \\ \tau/2 & \text{if } q_1 = q_2 \\ (q_2 - q_1)^{-1} & \text{if } q_1 > q_2, \end{cases}$$



$\mathbb{E}(T_1 | T > \tau)$ and $\mathbb{E}(T_2 | T > \tau)$ versus τ (with $q_1 < q_2$)

Conditional analysis, $n > 2$

Insights for $n = 2$ remain valid for $n > 2$. Let

$$\begin{aligned} \beta_1 &= \min\{q_j\} && \text{(slowest rate)} \\ r_1 &= \#j \text{ with } q_j = \beta_1 && \text{(\# of slowest states)} \\ \beta_2 &= \min\{q_j : q_j \neq \beta_1\} && \text{(second-slowest rate)} \end{aligned}$$

Then, for τ large, conditional sojourn time distribution of state j depends on q_j, and if $q_j = \beta_1$ also on r_1:

- $q_j > \beta_1$: $\sim exp(q_j - \beta_1)$
- $q_j = \beta_1, r = 1$: $\sim exp(\beta_1)$ for $t > \tau$, $\sim exp(\beta_2 - \beta_1)$ for $t < \tau$
- $q_j = \beta_1, r > 1$: $\sim exp(\beta_1)$ for $t > \tau$, polynomial for $t < \tau$

Conclusion for case (ii)

- Replace conditional density $f_j(t|T > \tau)$ of T_j by

$$f_j^*(t) = \begin{cases} (q_j - \beta_1) \cdot e^{-(q_j - \beta_1)\cdot t} & \text{if } q_j > \beta_1 \\ f_1^{(n=2)}(t|T > \tau)\Big|_{(q_1,q_2)=(\beta_1,\beta_2)} & \text{if } q_j = \beta_1, r_1 = 1 \\ r_1/\tau \cdot e^{-r_1/\tau\cdot t} & \text{if } q_j = \beta_1, r_1 > 1 \end{cases}$$

Results

- 10^6 simulation runs
- Standard Monte Carlo (MC) estimator $\hat{p}$ versus
- Importance Sampling (IS) estimator $\hat{p}^*$
- Compare r.e. = relative error$\times$1.96 (relative half-width of estimated 95% Conf. Int.)

$$n = 2, q_1 < q_2:$$

τ	$\hat{p}$	MC-r.e.	$\hat{p}^*$	IS-r.e.	true
5	2.52E-4	0.1235	2.417E-4	0.0047	2.417E-4
7	8.0E-6	0.6929	4.71E-6	0.0054	4.736E-6
9	0	—	8.947E-8	0.0060	8.93E-8
100	0	—	8.3E-89	0.0078	8.3E-89

Bounded relative error (?)

Results

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- Compare r.e. = relative error$\times$1.96 (relative half-width of estimated 95% Conf. Int.)

$$n = 2, q_1 = q_2:$$

τ	$\hat{p}$	MC-r.e.	$\hat{p}^*$	IS-r.e.	true
5	2.03E-4	0.1375	2.011E-4	0.0058	2.004E-4
7	3.0E-6	1.1316	3.372E-6	0.0070	3.363E-6
100	0	—	6.29E-94	0.0279	6.3E-94

(bit) less accurate, due to $f_1^*(t) \not\approx f_1(t|T > \tau)$ for $t < \tau$ (?)

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$$n = 50, q_i = \lceil \frac{i+1}{2} \rceil, i = 1, \ldots, 50:$$

τ	$\hat{p}$	MC-r.e.	$\hat{p}^*$	IS-r.e.	true
12	2.092E-2	0.0134	2.097E-2	0.0051	—
20	1.4E-5	0.5238	1.727E-5	0.0070	—
100	0	—	2.19E-39	0.0180	—

Conclusions

- Fast simulation for Slow paths is interesting (reaching target state *after* some large time bound)
- Importance Sampling helps...
- ... but not by exponential tilting
- Burden of reaching large time bound is (almost) only for slowest state(s)

Future work:

- Prove asymptotics for conditional distributions
- Investigate bounded relative error (?)
- Extend to general CTMC, i.e. sample appropriate paths

Thanks you for your attention!